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COMMON FIXED-POINTS OF NONSELF *I*-QUASI-NONEXPANSIVE MAPPINGS FOR MULTI-STEP ITERATION IN BANACH SPACES

JAMNIAN NANTADILOK

Department of Mathematics
Faculty of Science
Lampang Rajabhat University
Lampang, 52100
Thailand
e-mail: jamnian52@lpru.ac.th

Abstract

Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T : K \to E$ be a nonself mapping. In this paper, we establish the weak convergence of a sequence of a modified multistep iteration of a nonself I-quasi-nonexpansive mapping in a Banach space which satisfies Opial's condition.

1. Introduction

Let E be a normed linear space, K be a nonempty, convex subset of E, and T be a self map of K. Three most popular iteration procedures for obtaining fixed points of T, if they exist, are Mann iteration [8], defined by

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$$u_1 \in K,\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n,\ n \geq 1. \tag{1.1}$$

Ishikawa iteration [4], defined by

$$z_1 \in K,\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n)z_n + \beta_n T z_n,\ n \geq 1. \tag{1.2}$$

Noor iteration [9], defined by

$$v_1 \in K,\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n T w_n,$$
$$w_n = (1-\beta_n)v_n + \beta_n T t_n,$$
$$t_n = (1-\gamma_n)v_n + \gamma_n T v_n,\ n \geq 1, \tag{1.3}$$

for certain choices of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \subset [0, 1]$.

The multi-step iteration [12], arbitrary fixed order $p \geq 2$, defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1,$$
$$y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1},\quad i = 1, 2, \dots, p-2,$$
$$y_n^{p-1} = (1-\beta_n^{p-1}x_n + \beta_n^{p-1} T x_n), \tag{1.4}$$

where, for all $n \in N$,

$$\{\alpha_n\} \subset (0, 1),\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and for all $n \in N$,

$$\{\beta_n^i\} \subset [0, 1),\ 1 \leq i \leq p-1,\ \lim_{n\to\infty} \beta_n^i = 0.$$

Taking $p = 3$ in (1.4), we obtain iteration (1.3). Taking $p = 2$ in (1.4), we obtain iteration (1.2).

2. Preliminaries

Let K be a subset of normed linear space E and T be a self-mapping of K. Then T is called *nonexpansive on K*, if

$$\|Tx - Ty\| \le \|x - y\|, \tag{2.1}$$

for all $x, y \in K$. Let $F(T) := \{x \in K : Tx = x\}$ denote the set of fixed points of a mapping T.

Let K be a subset of normed linear space E, T and I be self-mappings of K. Then T is called *I-nonexpansive on K*, if

$$\|Tx - Ty\| \le \|Ix - Iy\|, \tag{2.2}$$

for all $x, y \in K$ [14]. T is called *I-quasi-nonexpansive* on K, if

$$\|Tx - f\| \le \|Ix - f\|, \tag{2.3}$$

for all $x \in K$ and $f \in F(T) \cap F(I)$.

Let E be a real Banach space. A subset K of E is said to be a retract of E, if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be retraction, if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. Recall that a Banach space E is said to satisfy Opial's condition [10] if, for each sequence $\{x_n\}$ in E, the condition $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \tag{2.4}$$

for all $y \in E$ with $y \ne x$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [1] and Dotson [2] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [5] in metric spaces, which we adapt to a normed space as follows: T is called a *quasi-nonexpansive mapping* provided that

$$\|Tx - f\| \le \|x - f\|, \tag{2.5}$$

for all $x \in K$ and $f \in F(T)$.

Remark 2.1. There are many results of fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshin and Williamson [11]. Their analysis was related to the convergence of Mann iterates studied by Dotson [2]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [3]. In [15], the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [16], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In [13], Rhoades and Temir considered T and I self-mappings of K, where T is an I-nonexpansive mapping and K be a nonempty closed convex subset of a uniformly convex Banach space. They established the weak convergence of the sequence of Mann iterates to a common fixed point of T and I. However, if the domain K of T is a proper subset of E and T maps K into E, then the iteration formula (1.1) may fail to be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retract $P : E \to K$ in the recursion formula (1.1) as follows: $u_1 \in K$,

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n PTu_n,\ n \geq 1.$$

In [6], Kiziltunc and Ozdemir considered T and I nonself-mappings of K, where T is an I-nonexpansive mapping. They established the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of T and I. In [7], Kiziltunc and Yildirim considered T and I nonself-mappings of K, where T is an I-nonexpansive mapping. They established the weak convergence of the sequence of modified multi-step iterative scheme $\{x_n\}$ defined by, arbitrary fixed order $p \geq 2$,

$$\begin{aligned} x_{n+1} &= P((1 - \alpha_n)x_n + \alpha_n Ty_n^1), \\ y_n^i &= P((1 - \beta_n^i)x_n + \beta_n^i Ty_n^{i+1}),\ i = 1, 2, \ldots, p-2, \\ y_n^{p-1} &= P((1 - \beta_n^{p-1})x_n + \beta_n^{p-1}Tx_n), \end{aligned} \tag{2.6}$$

where, for all $n \in N$,

$$\{\alpha_n\} \subset (0, 1), \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and for all $n \in N$,

$$\{\beta_n^i\} \subset [0, 1), 1 \leq i \leq p-1, \lim_{n\to\infty} \beta_n^i = 0.$$

Remark 2.2. Clearly, if T is a self-map, then (2.6) reduces to an iterative scheme (1.4).

In this paper, we consider T and I nonself mappings of K, where T is an I-quasi-nonexpansive mapping more general class of mappings than those mentioned in [7]. We establish weak convergence theorem of sequence of modified multi-step iterative scheme $\{x_n\}$ defined by (2.6) for nonself I-quasi-nonexpansive mapping T, where I is a quasi-nonexpansive mapping.

3. Main Results

Theorem 3.1. *Let K be a closed convex bounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T, I be nonself mappings of K with T an I-quasi-nonexpansive mapping, I a quasi-nonexpansive mapping on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified multistep iterates converges weakly to a common fixed point of $F(T) \cap F(I)$.*

Proof. If $F(T) \cap F(I)$ is nonempty and a singleton, then the proof is complete. We will assume that $F(T) \cap F(I)$ is not a singleton.

$$\begin{aligned}
\|x_{n+1} - f\| &= \|P((1-\alpha_n)x_n + \alpha_n Ty_n^1) - f\| \\
&\leq \|((1-\alpha_n)x_n + \alpha_n Ty_n^1) - f\| \\
&\leq \|(1-\alpha_n)(x_n - f)\| + \alpha_n\|T(P((1-\beta_n^1)x_n + \beta_n^1 Ty_n^2)) - f\| \\
&\leq \|(1-\alpha_n)(x_n - f)\| + \alpha_n\|I(P((1-\beta_n^1)x_n + \beta_n^1 Ty_n^2)) - f\|
\end{aligned}$$

$$\leq \|(1-\alpha_n)(x_n - f)\| + \alpha_n \|(P((1-\beta_n^1)x_n + \beta_n^1 T y_n^2)) - f\|$$

$$\leq \|(1-\alpha_n)(x_n - f)\| + \alpha_n \|(1-\beta_n^1)x_n + \beta_n^1 T y_n^2 - f\|$$

$$\leq \|(1-\alpha_n)(x_n - f)\| + \alpha_n \|(1-\beta_n^1)(x_n - f)\| + \alpha_n \beta_n^1 \|I(y_n^2) - f\|$$

$$\leq (1-\alpha_n)\|(x_n - f)\| + \alpha_n (1-\beta_n^1)\|(x_n - f)\| + \alpha_n \beta_n^1 \|y_n^2 - f\|$$

$$\leq (1-\alpha_n)\|(x_n - f)\| + \alpha_n (1-\beta_n^1)\|(x_n - f)\|$$

$$+ \alpha_n \beta_n^1 \|P((1-\beta_n^2)x_n + \beta_n^2 T y_n^3) - f\|$$

$$\leq (1-\alpha_n)\|(x_n - f)\| + \alpha_n (1-\beta_n^1)\|(x_n - f)\|$$

$$+ \alpha_n \beta_n^1 \|((1-\beta_n^2)x_n + \beta_n^2 T y_n^3) - f\|$$

$$\leq (1-\alpha_n)\|(x_n - f)\| + \alpha_n (1-\beta_n^1)\|(x_n - f)\|$$

$$+ \alpha_n \beta_n^1 \|(1-\beta_n^2)x_n + \beta_n^2 f - f + \beta_n^2 [T y_n^3 - f]\|$$

$$\leq [(1-\alpha_n) + \alpha_n(1-\beta_n^1) + \alpha_n \beta_n^1 (1-\beta_n^2)]\|x_n - f\|$$

$$+ \alpha_n \beta_n^1 \beta_n^2 \|I y_n^3 - f\|$$

$$\leq [(1-\alpha_n) + \alpha_n(1-\beta_n^1) + \alpha_n \beta_n^1 (1-\beta_n^2)]\|x_n - f\|$$

$$+ \alpha_n \beta_n^1 \beta_n^2 \|y_n^3 - f\|$$

$$\vdots$$

$$\leq [(1-\alpha_n) + \alpha_n(1-\beta_n^1) + \alpha_n \beta_n^1 (1-\beta_n^2)$$

$$+ \alpha_n \beta_n^1 \beta_n^2 (1-\beta_n^3) + \alpha_n \beta_n^1 \beta_n^2 \beta_n^3 (1-\beta_n^4)$$

$$+ \cdots + \alpha_n \beta_n^1 \beta_n^2 \beta_n^3 \ldots \beta_n^{p-3}(1-\beta_n^{p-2})]\|x_n - f\|$$

$$+ \alpha_n \beta_n^1 \beta_n^2 \beta_n^3 \ldots \beta_n^{p-2} \|y_n^{p-1} - f\|$$

$$\leq [(1-\alpha_n) + \alpha_n(1-\beta_n^1) + \alpha_n \beta_n^1 (1-\beta_n^2)$$

$$+ \alpha_n \beta_n^1 \beta_n^2 (1-\beta_n^3) + \alpha_n \beta_n^1 \beta_n^2 \beta_n^3 (1-\beta_n^4)$$

$$+ \cdots + \alpha_n \beta_n^1 \beta_n^2 \beta_n^3 \ldots \beta_n^{p-1}(1 - \beta_n^p)$$

$$+ \alpha_n \beta_n^1 \beta_n^2 \beta_n^3 \ldots \beta_n^p]\|x_n - f\|$$

$$= \|x_n - f\|. \tag{3.1}$$

Thus, for $\alpha_n \neq 0$ and $\beta_n^i \neq 0$, $\{\|x_n - f\|\}$ is a non-increasing sequence. Then $\lim_{n\to\infty}\|x_n - f\|$ exists.

Now, we show that $\{x_n\}$ converges weakly to a common fixed point of T and I. The sequence $\{x_n\}$ contains a subsequence which converges weakly to the point in K. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converges weakly to f and q, respectively. We will show that $f = q$. Suppose that E satisfies Opial's condition and that $f \neq q$ is in weak limit set of the sequence $\{x_n\}$. Then $x_{n_k} \rightharpoonup f$ and $x_{m_k} \rightharpoonup q$, respectively. Since $\lim_{n\to\infty}\|x_n - f\|$ exists for any $f \in F(T) \cap F(I)$, by Opial's condition, we conclude that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - f\| &= \lim_{k\to\infty}\|x_{n_k} - f\| \\
&< \lim_{k\to\infty}\|x_{n_k} - q\| \\
&= \lim_{j\to\infty}\|x_{m_j} - q\| \\
&< \lim_{j\to\infty}\|x_{m_j} - f\| \\
&= \lim_{n\to\infty}\|x_n - f\|.
\end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T) \cap F(I)$. □

From Theorem (3.1), we obtain the following corollary.

Corollary 3.2 (Kiziltunc and Yildirim [7]). *Let K be a closed convex bounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T, I be nonself mappings of K with T an I-*

nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, *the sequence* $\{x_n\}$ *of modified multi-step iterates converges weakly to a common fixed point of* $F(T) \cap F(I)$.

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